02 - Math Preliminaries

Dr. Robert Lowe

Division of Mathematics and Computer Science
Maryville College

Outline

Relation

- A **relation**, R between two sets, A and B is:

$$R \subset A \times B$$

- For example, consider the $<$ relation:

$$\begin{aligned}
\textbf{Let } S = & T = \{1, 2, 3\} \\
S \times T = & \{(1,1),(1,2),(1,3), \\
& (2,1),(2,2),(2,3), \\
& (3,1),(3,2),(3,3)\} \\
U = & \{(1,2),(1,3),(2,3)\}
\end{aligned}$$

  Thus U is the sought after $<$ relation!
- We often write relations this: aUb where aUb is true if and only if $(a, b) \in U$

Digraphs

A **digraph** is a graphical representation of a relation.

Example 2.3.1

The relationship 'is a town in' between S = the set of towns and T = the set of countries looks like this:

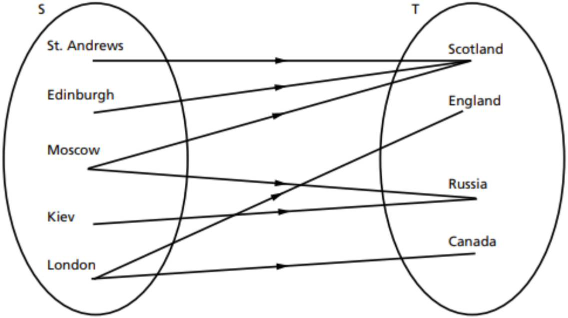

Fig. 2.1

From Davie and Morrison page 26

Properties of Relations

- A **includes** B if, $\forall s \in S$ and $t \in T$, $sBt \implies sAt$
- A is the **transpose** of B if, $\forall s \in S$ and $t \in T$, $sAt \iff tBs$
- A is **reflexive** if $S = T$ and $\forall s \in S$, sAs is true.
- A is **transitive** if $S = T$ and $\forall r, s, t \in R, S, T$, $rAs \text{ and } sAt \implies rAt$

Exercise

For each of the above properties, define a relation that exhibits this property.

Algebra of Relations

- Let A be a relation between R and S. ($A \subset R \times S$)
- Let B be a relation between S and T. ($B \subset S \times T$)
- The **product** $AB \subset R \times T$ is defined as:
  $rABt \iff \exists s \in S : rAs \text{ and } sBt$
- This product is associative but not commutative.
- The equality relation, I, is the identity

$$I_s A = A = A I_t$$

- We define **powers** of A as:
  - $A^0 = I$
  - $A^{n+1} = A^n A$ for $n > 0$

Transitive Closure

- let $A \subset S \times S$
- The transitive closure of A is:

$$A^{+} = \sum_{i=1}^{\infty} A^{i}$$

- Finite closures exist when A^{+} converges.

Reflexive Transitive Closure

Adding $I = A^0$ to A^+ yields the reflexive transitive closure of A.

$$A^* = \sum_{i=0}^{\infty} A^i$$

Example: Direct Divisor Relation

- Let $S = T =$ the set of divisors of 12.
- The direct divisor relation is $\{(1,2),(1,3),(2,4),(2,6),(3,6),(4,12),(6,12)\}$
- This relation's digraph is:

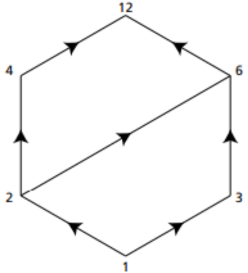

Fig. 2.4

(Davie and Morrison page 30)

Example: Binary Matrix of Direct Divisor Relation

M(A) =

	1	2	3	4	6	12
1		*	*			
2				*	*	
3					(*)	
4						*
6						*
12						

(Davie and Morrison page 30)

Adding Binary Matrix Representations of Relations

- $M(A + B) = M(A) \vee M(B)$
- Let's compute $M(A^2)$
- Next, compute $M(A + A^2)$
- Let's compute $A + A^2 + \ldots$ until it "settles down"
- This gives us A^+, to get A^* we just make all diagonal elements true.

Computing transitive closure in S-Algol

```
!Read or calculate bool matrix A(nxn)
for i=1 to n do     !Each time round, add a new node
                    !(number i) to the transitive closure graph
  for j=1 to n do   !Find all arrows leading into node i
  if A(j,i) do      !If there is an arrow from node j to node i,
                    !find all those out of node i
    for k=1 to n do !make a direct path arrow between nodes j
        A(j,k):=A(j,k) or A(i,k)
                    !Now A contains the transitive closure matrix
```

From Davie and Morrison page 33