A Translation of G. Cantor's "Ueber eine elementare Frage der Mannigfaltigkeitslehre"

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[1]*https://translate.google.com*
[2]*https://www.deepl.com*

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An English translation of G. Cantor's "Ueber eine elementare Frage der Mannigfaltigkeitslehre"[1] article: "On an elementary question of the theory of manifolds."

Translation Note: We have translated "Inbegriff" as collection, and "Mächtigkeit" as power. Apart from these adjustments and a few other specific edits the bulk of this English language text was obtained directly from the machine translators acknowledged as the main authors.

I.

In the essay titled "On a Property of the Collection of All Real Algebraic Numbers" (Journ. Math. Vol. 77, p.258), it is probable that for the first time there is proof for the proposition that there are infinite manifolds which do not relate to each other uniquely to the set of all finite integers $1, 2, 3, \ldots, \nu, \ldots$ or, as I say, that do not have the power of the $1, 2, 3, \ldots, \nu, \ldots$ series.

As shown there in Sec.2, it follows without further ado that, for example, the totality of all real numbers of any interval (a, b) can not be imagined in the series form:

$$\omega_1, \omega_2, \ldots, \omega_\nu, \ldots$$

But it is possible to provide a much simpler proof of that theorem, which is independent of the consideration of irrational numbers.

If m and w are any two mutually exclusive characters, then we consider a collection M of elements, $E = (x_1, x_2, \ldots, x_\nu, \ldots)$, which depend on an infinite number of coordinates, $x_1, x_2, \ldots, x_\nu, \ldots$, where each of these coordinates is either m or w.

M is the totality of all elements E.

The elements of M include, for example, the following three:

$$E^I = (m, m, m, m, \ldots),$$
$$E^{II} = (w, w, w, w, \ldots),$$
$$E^{III} = (m, w, m, w, \ldots).$$

I now claim that such a manifold M does not have the power of the series $1, 2, 3, \ldots, \nu, \ldots$.

This follows from the following sentence:

> If $E_1, E_2, \ldots, E_\nu, \ldots$ are any simply infinite series of elements of the manifold M, then there is always an element E_0 of M that does not agree with any E_ν.

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To prove it:

$$\begin{aligned} E_1 &= (a_{1,1}, a_{1,2}, \ldots, a_{1,\nu}, \ldots), \\ E_2 &= (a_{2,1}, a_{2,2}, \ldots, a_{2,\nu}, \ldots), \\ &\ldots \\ E^\mu &= (a_{\mu,1}, a_{\mu,2}, \ldots, a_{\mu,\nu}, \ldots). \\ &\ldots \end{aligned}$$

Here the $a_{\mu,\nu}$ are in a certain way m or w. Let us now define a series $b_1, b_2, \ldots, b_\nu, \ldots$ such that b_ν is also only equal to m or w and different from $a_{\nu,\nu}$.

So if $a_{\nu,\nu} = m$, then $b_\nu = w$, and if $a_{\nu,\nu} = w$, then $b_\nu = m$.

If we then consider the element:

$$E_0 = (b_1, b_2, b_3, ...)$$

of M, we can easily see that the equation:

$$E_0 = E_\mu$$

for no positive integer value of μ, can be satisfied, otherwise for the given μ and for all integer values of ν:

$$b_\nu = a_{\mu,\nu},$$

so also in particular,

$$b_\mu = a_{\mu,\mu},$$

which would be excluded by the definition of b_ν.

From this theorem follows immediately that the totality of all elements of M can not be put into the series form: $E_1, E_2, \ldots, E_\nu, \ldots$, otherwise we would be faced with the contradiction that a thing E_0 is both an element of M as well as not being element of M.

This proof is noteworthy not only for its great simplicity, but also for the reason that the principle followed therein can be readily extended to the general proposition that the powers of well-defined manifolds have no maximum or, what is the same, that for any given manifold L another M can be put on the side, which is of greater power than L.

For example, let L be a linear continuum, for example, the collection of all real number numbers z, which are $\geqslant 0$ and $\leqslant 1$.

Under M we understand the collection of all unique functions $f(x)$ which only take the two values 0 or 1, while x passes through all real values which are $\geqslant 0$ and $\leqslant 1$.

The fact that M has no smaller power than L, follows from the fact that subsets of M can be specified, which have the same power as L, for example, the subset consisting of all the functions of x that have the value 1 for a single value x_0 of x, and 0 for all other values of x.

But also M does not have the same power with L, because otherwise the manifold M could be brought into a mutually unambiguous relation to the variable z, and M could be thought of in the form of an unambiguous function of the two variables x and z: $\phi(x, z)$, so that through each specialization of z an element $f(x) = \phi(x, z)$ of M is obtained and also vice versa each element $f(x)$ of M emerges from $\phi(x, z)$ through a single particular specialization of z. However, this leads to a contradiction.

For if we understand $g(x)$ to be that unique function of x which takes only the values 0 or 1 and is different for every value of x of $\phi(x,x)$, then on the one hand $g(x)$ is an element of M, on the other hand, $g(x)$ can not result from $\phi(x,z)$ by any specialization $z = z_0$ because $\phi(z_0, z_0)$ is different from $g(z_0)$.

If, therefore, the power of M is neither smaller nor equal to that of L, it follows that it is greater than the power of L. (Cf. Crelle's Journal, vol. 84, p. 242).

Already in the "Foundations of a General Theory of Manifolds" (Leipzig, 1883, Math. Annalen, Vol. 21), I have shown by very different means that the powers have no maximum; there it was even proved that the collection of all powers, if we think of the latter in order of size, forms a "well-ordered crowd", so that in nature there is one next greater in every power, but also a next greater one follows every infinitely increasing set of powers.

The "powers" represent the only and necessary generalization of the finite "cardinal numbers", they are nothing else than the actual infinite-sized cardinal numbers, and they have the same reality and certainty as those; only that the lawful relations among them, the "number theory" related to them is partly different, than in the area of the finite.

The further development of this field is the task of the future.

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[1] Georg Cantor. Ueber eine elementare Frage der Mannigfaltigkeitslehre. *Jahresbericht der Deutschen Mathematiker-Vereinigung*, 1:75–78, 1891.